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LETTER TO THE EDITOR

Potts model on finitely ramified fractals

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Abstract. We present an exact solution for the ferromagnetic q-state Potts model on a family of finitely ramified fractal lattices with fractal dimension between 1 and 2. We obtain the thermal exponent ν for the whole fractal family. The results stress the connection between geometrical parameters and critical behaviour, opening up new possibilities for modelling physical systems showing fractal structures.

The interest in studying phenomena involving fractal structures (Mandelbrot 1977) has increased in recent years. For example, recent experimental works on kinetic gelation (Weitz and Oliveria 1984, Schaefer *et al* 1984) and on two-dimensional growth of metal clusters (Matsushita *et al* 1984) reveal that aggregates are well described by fractals with Hausdorff dimensions determined with high accuracy. Also, self-similar fractal structures (Mandelbrot 1977) have been found on percolation clusters (Leath and Reich 1978, Stanley 1977, Stauffer 1979, Pike and Stanley 1981, Gaunt and Sykes 1983). Their full geometrical characterisation is important in understanding the thermal properties of random spin systems near the percolation threshold, through the behaviour of related spin systems on the backbone geometry of percolating clusters. In particular, one hopes that a systematic study of spin systems on fractal lattices will be useful in modelling the backbone.

In this letter we study the Potts model on a family of self-similar fractals embedded in two-dimensional Euclidean space. These fractals have a finite order of ramification and provide intermediate geometries between the quasi-one-dimensional structure (e.g. Koch curves) and infinitely ramified fractals (e.g. Sierpinski carpets) (Mandelbrot 1977). The geometrical properties of the fractals studied here have been discussed by Hilfer and Blumen (1984). They are obtained iteratively from generators G_n which, in turn, are built up from elementary triangular structures (see figure 1). We define the linear dimension of G_n as $b=3n-1$.

The structure generated at each step of the iterative procedure is obtained by replacing all upward pointing triangles by a G_n structure. This procedure goes on indefinitely. The Sierpinski gasket is the special case $n=1$ ($b=2$). Figure 2 displays the $n=2$ ($b=5$) case, after one step of the iterative procedure.

An important geometrical parameter is the Hausdorff dimension. For the fractal generated by G_n, it is given by (Hilfer and Blumen 1984):

$$d_f = \ln N / \ln b \tag{1}$$

where N is the number of small upward pointing triangles inside G_n and b is the linear dimension of G_n; clearly one has $1 < d_f < 2$.

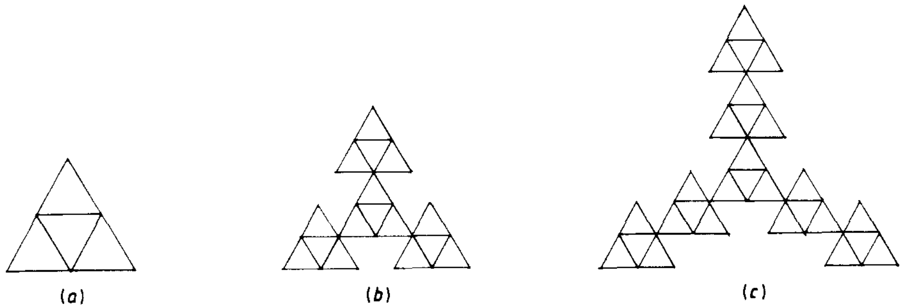

Figure 1. G_n generators (*a*) $n=1$, (*b*) $n=2$, (*c*) $n=3$.

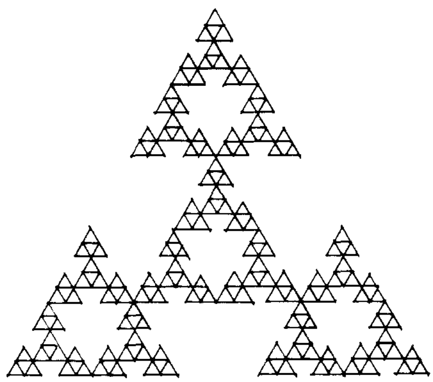

Figure 2. Resulting structure after one step in the iterative construction of the fractal lattice generated by G_2.

At each site of the fractals defined above one attaches a spin σ_i, which can be in any of the q Potts states. The Potts Hamiltonian then reads (in units of $1/\beta = k_B T$):

$$-\beta\mathscr{H} = \sum_{\langle ij\rangle} J(\delta_{\sigma_i\sigma_j} - 1) + \sum_{\langle ij\rangle} M(\delta_{\sigma_i\sigma_j} - 1)(\delta_{\sigma_j\sigma_k} - 1)(\delta_{\sigma_k\sigma_i} - 1) \qquad (2)$$

where $\langle ij\rangle$ and $\langle ijk\rangle$ stand for first-neighbour pairs and basic triangles, respectively, after a microscopic scale is reached.

Due to their finite order of ramification, spin systems on these lattices are exactly solvable. Generally the renormalisation group (RG) equations are obtained from $Z_n(\sigma_{A_n}\sigma_{B_n}\sigma_{C_n})$, the partial trace over all internal spins of G_n, keeping the spins on vertices A_n, B_n and C_n at fixed states σ_{A_n}, σ_{B_n} and σ_{C_n} respectively (see figures 3(*a*) and 3(*b*)).

The resulting RG transformations are of the form

$$e^{-2J'} = Z_n(\alpha\alpha\beta)/Z_n(\alpha\alpha\alpha) \qquad e^{-2J'-M'} = Z_n(\alpha\beta\gamma)/Z_n(\alpha\alpha\alpha) \qquad (3a, b)$$

where α, β, γ denote different Potts states.

The internal summation becomes cumbersome as b increases, but it can be performed recursively between consecutive generators G_n. Consider for example the generator G_2. The summation of its internal configurations can be decoupled into independent sums over four graphs by keeping fixed the spins attached to their intersections (vertices A_1, B_1 and C_1 in figure 3(*a*)). The sum in the central graph is

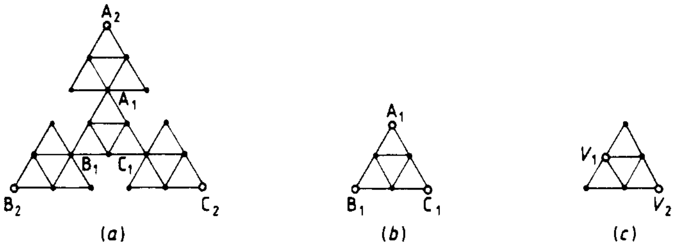

Figure 3. Illustration of the summation of internal configurations in G_2 through the decomposition into four graphs: sites with fixed spins are denoted by ○ (*a*) $Z_2(\sigma_{A_2}\sigma_{B_2}\sigma_{C_2})$, (*b*) $Z_1(\sigma_{A_1}\sigma_{B_1}\sigma_{C_1})$, (*c*) $S(\sigma_{V_1}\sigma_{V_2})$.

just $Z_1(\sigma_{A_1}\sigma_{B_1}\sigma_{C_1})$ (figure 3(*b*)) and the sums inside the other three graphs are called $S(\sigma_{V_1}\sigma_{V_2})$, where σ_{V_1} and σ_{V_1} stand for fixed spins on vertices V_1 and V_2, V = A, B or C (see figure 3(*c*)).

The result can be written as

$$Z_2(\sigma_{A_2}\sigma_{B_2}\sigma_{C_2}) = \sum_{\sigma_{A_1}\sigma_{B_1}\sigma_{C_1}} Z_1(\sigma_{A_1}\sigma_{B_1}\sigma_{C_1})S(\sigma_{A_1}\sigma_{A_2})S(\sigma_{B_1}\sigma_{B_2})S(\sigma_{C_1}\sigma_{C_2}). \tag{4}$$

Equation (4) can be generalised for consecutive generators G_n and G_{n+1} as

$$Z_{n+1}(\sigma_{A_{n+1}}\sigma_{B_{n+1}}\sigma_{C_{n+1}}) = \sum_{\sigma_{A_n}\sigma_{B_n}\sigma_{C_n}} Z_n(\sigma_{A_n}\sigma_{B_n}\sigma_{C_n})S(\sigma_{A_n}\sigma_{A_{n+1}})S(\sigma_{B_n}\sigma_{B_{n+1}})S(\sigma_{C_n}\sigma_{C_{n+1}}). \tag{5}$$

The finite order of ramification of the lattices implies $T_c = 0$, a result that can be understood from entropy arguments (Gefen *et al* 1984a). To obtain the low-temperature behaviour ($J, M \gg 1$) it is only necessary to calculate the energy of the lowest lying excitations.

In the low-temperature limit, the dominant contributions to (5) are given by

$$Z_{n+1}(\alpha\alpha\alpha) = Z_n(\alpha\alpha\alpha)S^3(\alpha\alpha) + 3(q-1)Z_n(\alpha\alpha\beta)S^2(\alpha\alpha)S(\alpha\beta) \tag{6a}$$

$$Z_{n+1}(\alpha\alpha\beta) = Z_n(\alpha\alpha\alpha)S^2(\alpha\alpha)S^2(\alpha\beta) + Z_n(\beta\beta\beta)S(\alpha\alpha)S^2(\beta\beta) + Z_n(\alpha\alpha\beta)S^3(\alpha\alpha) + (q-2)Z_n(\alpha\alpha\gamma)S^2(\alpha\alpha)S(\alpha\beta) + 2Z_n(\alpha\beta\beta)S^2(\alpha\alpha)S(\alpha\beta) \tag{6b}$$

$$Z_{n+1}(\alpha\beta\gamma) = 3Z_n(\alpha\alpha\alpha)S(\alpha\alpha)S^2(\alpha\beta) + 6Z_n(\alpha\alpha\beta)S^2(\alpha\alpha)S(\alpha\beta) + Z_n(\alpha\beta\gamma)S^3(\alpha\alpha) \tag{6c}$$

where the permutation symmetry in $S(\sigma\sigma') = S(\sigma'\sigma)$ between Potts states was used.

One has, to leading order in e^{-J} and e^{-M}

$$Z_1(\alpha\alpha\alpha) = 1 + 3(q-1)\,e^{-4J} + 4(q-1)\,e^{-6J} \tag{7a}$$

$$Z_1(\alpha\alpha\beta) = e^{-2J} + 4e^{-4J} + 2(q-2)\,e^{-5J-M} + 3(q-1)\,e^{-6J} \tag{7b}$$

$$Z_1(\alpha\beta\gamma) = 3e^{-4J} + 6e^{-5J-M} + (q+3)\,e^{-6J} \tag{7c}$$

which reproduces the results of Gefen *et al* (1984a) for the Sierpinski gasket lattice, and

$$S(\alpha\alpha) = 1 + 2(q-1)\,e^{-2J} + (q-1)(q+3)\,e^{-4J} + 2(q-1)(q-2)\,e^{-5J-M} \tag{8a}$$

$$S(\alpha\beta) = e^{-2J} + 2(q+3)\,e^{-4J} + 6(q-2)\,e^{-5J-M}. \tag{8b}$$

Combining expressions (6), (7) and (8) we finally obtain

$$Z_2(\alpha\alpha\alpha) = 1 + 6(q-1)\,e^{-2J} + 3(q-1)(q+5)\,e^{-4J} + 6(q-1)(q-2)\,e^{-5J-M} \tag{9a}$$

$$Z_2(\alpha\alpha\beta) = 2e^{-2J} + (13q+1)\,e^{-4J} + 8(q-2)\,e^{-5J-M} \tag{9b}$$

$$Z_2(\alpha\beta\gamma) = 12e^{-4J} + 6e^{-5J-M} \tag{9c}$$

Analogously, combining (6), (8) and (9):

$$Z_3(\alpha\alpha\alpha) = 1 + 12(q-1)\,e^{-2J} + 6(q-1)(q+5)\,e^{-4J} + 12(q-1)(q-2)\,e^{5J-M} \tag{10a}$$

$$Z_3(\alpha\alpha\beta) = 3e^{-2J} + (39q-14)\,e^{-4J} + 14(q-2)\,e^{-5J-M} \tag{10b}$$

$$Z_3(\alpha\beta\gamma) = 27e^{-4J} + 6e^{-5J-M}. \tag{10c}$$

Using (9), the RG equations (3) for the $n=2$ fractal in the limit $J, M \gg 1$, become

$$e^{-2J'} = 2e^{-2J} + a\,e^{-4J} + 8(q-2)\,e^{-5J-M} \tag{11a}$$

$$e^{-3J'-M'} = 12e^{-4J} + 6e^{-5J-M} \tag{11b}$$

with $a = 2(q+3) + 10(q-1) + (q+5)$.

Choosing $x = e^{-2J}$ and $y^2 = e^{-3J-M}$ as the low-temperature variables, equations (11) are rewritten, to second order in x and y, as

$$x' = 2x + ax^2 + 8(q-2)xy^2 \tag{12a}$$

$$y' = 2\sqrt{3}x + \tfrac{1}{2}\sqrt{3}y^2. \tag{12b}$$

In the same way the RG equations (3) for the $n=3$ fractal become

$$x' = 3x + bx^2 + 14(q-2)xy^2 \tag{13a}$$

$$y' = 3\sqrt{3}x + \tfrac{1}{3}\sqrt{3}y^2 \tag{13b}$$

with $b = 4(q+3) + 32(q-1) + 3(q+2)$.

By the recurrent formation of the coefficients of $Z_n(\sigma_{A_n}\sigma_{B_n}\sigma_{C_n})$ in (6), it is possible to write down the linearised RG equations in matrix form near the $T=0$ fixed point; for the lattice generated by G_n, one has

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} n & 0 \\ n\sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{14}$$

which yields eigenvalues n and 0 with eigenvectors $\binom{1}{\sqrt{3}}$ and $\binom{0}{1}$, respectively.

The variable y is then highly irrelevant, associated with a zero eigenvalue. From the eigenvalue $\lambda = n$, the thermal scaling power y_T is given by

$$y_T = \ln n / \ln(3n+1). \tag{15}$$

Table 1 shows the values of y_T for several lattices. As the fractal dimension (1) decreases, the thermal exponent $\nu = y_T^{-1}$ also decreases towards the limiting value $\nu = 1$ as d_f approaches one. This monotonic decrease of the thermal exponent ν with d_f differs from that of infinitely ramified fractals, where ν increases when d_f decreases and which also depends on other geometrical parameters apart from the fractal dimensionality (Gefen *et al* 1984b, Riera and Chaves 1985). It would be very interesting

Table 1. Fractal dimension and thermal scaling power for several lattices generated by G_n. $b=3n-1$, $d_f=\ln(9n-6)/\ln b$ and $Y_T=\ln n/\ln b$ (see text).

n	d_f	y_T
1	1.585	0
2	1.544	0.431
3	1.464	0.528
4	1.418	0.578
5	1.388	0.610
8	1.336	0.663
15	1.277	0.716
30	1.242	0.757
100	1.192	0.808

if exact solutions were available for fractals with different degree of ramification, so that the influence of geometrical properties on the critical exponents could be assessed.

For the lattices studied here, a universal behaviour between systems with different number of Potts states is obtained unlike the case of infinitely ramified fractal lattices (Riera and Chaves 1985). The possibility of universal behaviour must be investigated for other discrete spin models on finitely ramified fractals.

The members with $n \neq 1$ of the fractal family studied here provide intermediate geometries between two models discussed in the literature for the backbone of the infinite percolation cluster at p_c. The first one is the Sierpinski gasket model which is formed by interconnected loops only and gives good results for the fractal dimensionality of the backbone at low dimensions (Gefen *et al* 1981) but fails to predict the thermal behaviour of dilute spin systems at the percolation threshold. On the other hand, while the nodes, links and blobs model (an alternate sequence of singly connected and multiconnected bonds) yields the exact thermal-geometrical crossover exponent for any dimension (Coniglio 1982), the dominant role of singly connected bonds in the thermal transition at p_c is not fully understood, at least for low dimensionalities.

For the whole fractal family, the relevant geometrical feature in the determination of the thermal exponent is the presence of multiconnected links. Equation (15) relates ν with the index n of the generator G_n (this index can be associated with the number of G_1 vertices appearing in the G_n structure).

The $n \neq 1$ fractals also provide more reliable values for the exponent ν than the Sierpinski gasket model for the backbone. Thus, one has greater flexibility in the choice of a fractal structure as a model for the percolating cluster.

In summary, we presented an exact solution for the q-state Potts model on a family of finitely ramified lattices in which $1 < d_f < 2$. For each lattice, the critical behaviour was the same irrespective of the number of Potts states, in contrast to the results for infinitely ramified lattices. For the $n = 1$ fractal lattice, the results of Gefen *et al* (1984a) for the Potts model on the Sierpinski gasket were reproduced. The solutions for the $n \neq 1$ members of the family differ qualitatively from the Sierpinski gasket in the non-marginal character of the temperature. These results contribute to a more general view of the influence of geometrical mechanisms on critical behaviour, offering new possibilities to model general clustering properties of spin systems.

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